Abstract

The following notes are some “excerpts” from a course on particle physics, taught by Professor Rouven Essig in Fall 2016 at Stony Brook University. These notes aren’t at all complete - I’ve attempted to focus on the topics that are somewhat less standard in a typical particle physics class, and have more of a modern outlook (with a significant bias towards my own interests). I’ve also attempted to expand the notes and add interesting references where possible. Any errors in these notes were introduced in my transcription, and are no fault of the (wonderful) lecturer.

Coming Soon: Updated section on electroweak phenomenology and precision tests, section on CP violation and flavor physics, and experimental bounds on the neutrino masses and hierarchy.
### 3.3 Other Limits on the Higgs Mass
- 3.3.1 The Triviality Bound
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- 4.1 Yukawa Interactions
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  - 5.4.3 Oscillation Parameters
  - 5.4.4 The Mass Hierarchy
1 Introduction and Overview of the Standard Model

The following are typed notes taken during a course on particle physics taught by Professor Rouven Essig in Fall of 2016 at Stony Brook University. The course was an introductory course in particle physics, taught to both theory and experimental students, but which assumed a background in quantum field theory. Unfortunately it’s impossible to cover the breadth of particle physics in a one semester course, and this class instead took the route of digging into a subset of topics in more detail. Unfortunately, these notes contain even less than we covered in class – since a field theory background was essential, I’ve left out many of the lectures from the early part of the class, which reviewed Lagrangian field theory, symmetries and Lie groups, non-abelian gauge invariance, and some basics of spinors. I also left out the brief overview of QCD and it’s phenomenology - perhaps this will be remedied in the future.

The primary text referenced throughout the course was Langacker’s book, The Standard Model and Beyond [1]. Numerous other quantum field theory references were invaluable, particularly Peskin & Schroeder [2] and Schwartz’s more recent book [3].

There are a number of great references for Higgs physics, especially concerning the LHC. Some of the author’s favorites are [4] and [5]. For a very brief introduction to the physics behind the LHC program, see [6]. Finally, for a wealth of details on neutrino physics, one could check out the book by Zuber [7].

In the rest of this section, I review the make-up of the standard model, in large part to set conventions for the rest of the notes, but also to quickly cover electroweak symmetry breaking and provide some context for the rest of the course. In Section 2, we cover some implications of unitarity and it’s importance in particle physics. In Section 3 we review the phenomenology of the Higgs and collider searches. Section 4 discusses the Yukawa sector of the standard model, including both CP violation and flavor physics. Finally, Section 5 is the only section on BSM physics – covering neutrino masses and oscillations.

1.1 The Standard Model

The standard model of particle physics describes the interactions between the most fundamental constituents of the universe that we know of, on scales approximately $10^{-18}$ meters. To our best knowledge, it describes the universe all the way back to around a second after the big bang, and does remarkably well at describing almost all the data at particle colliders to date.

The content of the standard model can be broken down into matter particles and forces governed by gauge interactions with $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$. The matter content is shown below in Table 1.1, while the force mediating particles are shown in Table 1.1.

1.1.1 Formal Structure

The Standard Model is most easily described via Lagrangian quantum field theory. In order to write the Lagrangian, we write the matter content as a set of multiplets labelled
<table>
<thead>
<tr>
<th>Name</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>0.002</td>
</tr>
<tr>
<td>d</td>
<td>0.005</td>
</tr>
<tr>
<td>c</td>
<td>1.3</td>
</tr>
<tr>
<td>s</td>
<td>0.1</td>
</tr>
<tr>
<td>t</td>
<td>172.5</td>
</tr>
<tr>
<td>b</td>
<td>4.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>0.000511</td>
</tr>
<tr>
<td>$\nu_e$</td>
<td>$\lesssim 10^{-11}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.105</td>
</tr>
<tr>
<td>$\nu_\mu$</td>
<td>$\sim 0$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.8</td>
</tr>
<tr>
<td>$\nu_\tau$</td>
<td>$\sim 0$</td>
</tr>
</tbody>
</table>

Table 1: The fermion matter content of the standard model, with the particle masses shown. Quarks are listed as up- then down-type in each generation. The neutrinos are massless in the SM.

<table>
<thead>
<tr>
<th>Force</th>
<th>Mediator</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>gluons, $g$</td>
<td>0</td>
</tr>
<tr>
<td>Weak</td>
<td>$Z^0$, $W^\pm$</td>
<td>91, 80</td>
</tr>
<tr>
<td>EM</td>
<td>photon, $\gamma$,</td>
<td>0</td>
</tr>
<tr>
<td>Higgs</td>
<td>$\phi$ (or $H$)</td>
<td>125</td>
</tr>
</tbody>
</table>

Table 2: Forces and force carriers of the standard model, with the mass of the mediating particles.
\[ Q_L, u_R, d_R, L_L, e_R, \text{ where} \]
\[ Q_{Li} = \left( \begin{array}{c} u_L \\ d_L \\ \nu_L \\ e_L \end{array} \right)_i, \quad L_{Li} = \left( \begin{array}{c} u_R \\ d_R \\ \nu_R \\ e_R \end{array} \right)_i \]

(1.1)

and the index \( i = 1, 2, 3 \) runs over the three generations in the SM. We can summarize the charge under each gauge group as follows:

\[ Q_{Li}(3, 2)_{+1/6}, \quad u_{Ri}(3, 1)_{+2/3}, \quad d_{Ri}(3, 1)_{-1/3}, \quad L_{Li}(1, 2)_{-1}, \quad e_{Ri}(1, 1)_{-1} \]

(1.2)

where, for example, \( Q_L \) is a triplet under \( SU(3)_c \), a doublet under \( SU(2)_L \), and has hypercharge \( Y = +1 = \frac{1}{6} \). After spontaneous symmetry breaking, the electric charge of each field is given by \( Q_{EM} = T^3 + Y \), e.g., \( Q_{EM}(u_L) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \).

In addition to the fermion multiplets above, there is one complex scalar multiplet representing the Higgs:

\[ \phi(1, 2)_{+1/2} \]

(1.3)

which is sometimes also represented by \( H \).

The most general renormalizable Lagrangian for the standard model can be written as follows:

\[ \mathcal{L}_{SM} = \mathcal{L}_{kin} + \mathcal{L}_\phi + \mathcal{L}_{yuk} \]

(1.4)

where \( \mathcal{L}_{kin} \) is the kinetic term for all the matter and gauge bosons, \( \mathcal{L}_\phi \) is the scalar potential, and \( \mathcal{L}_{yuk} \) describes the Yukawa couplings between the fermionic matter and scalars. Note that there is no explicit mass term for the fermions – the origin of fermion mass via electroweak symmetry breaking will be described in the next section. To describe the gauge forces in \( G_{SM} \) we need generators \( L_a \) \((a = 1, \ldots, 8)\) for \( SU(3)_c \), and \( T_b \) \((b = 1, 2, 3)\) for \( SU(2)_L \). The generators satisfy the commutation relations dictated by their Lie algebras:

\[ [L_a, L_b] = if_{abc}L_c, \quad [T_a, T_b] = i\epsilon_{abc}T_c \]

(1.5)

The gauge forces are mediated by gauge bosons \( G^\mu_a \), \( W^\mu_b \), and \( B^\mu \) for \( SU(3)_c \), \( SU(2)_L \), and \( U(1)_Y \) respectively. It’s convenient to write these in terms of the gauge-invariant field strengths,

\[ G^{\mu\nu}_a = \partial^\mu G^\nu_a - \partial^\nu G^\mu_a - g_s f_{abc} G^\mu_b G^\nu_c \]

\[ W^{\mu\nu}_a = \partial^\mu W^\nu_a - \partial^\nu W^\mu_a - g_{abc} W^\mu_b W^\nu_c \]

\[ B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \]

(1.6) (1.7) (1.8)

where \( g_s, g \) and \( g' \) are the couplings of the strong and electroweak forces respectively.

To describe the fermion kinetic terms, we write a general covariant derivative as

\[ D_\mu = \partial_\mu + ig_S G^\mu_a L_a + ig W^\mu_b T_b + ig' Y B^\mu. \]

(1.9)

where the coefficients depend on the charge of the multiplet being acted on, e.g.,

\[ D^\mu Q_L = \left( \partial^\mu + i\frac{g_S}{2} G^\mu_a \lambda_a + i\frac{g}{2} W^\mu_a \sigma^a + i\frac{1}{6} g' B^\mu \right) Q_L, \]

\[ D^\mu \phi = \left( \partial^\mu + i\frac{g}{2} W^\mu_a \sigma_a + i\frac{1}{2} g' B^\mu \right) \phi \]
where we’ve introduced the standard Gell-Mann matrices \( \lambda_a \) and the Pauli matrices \( \sigma_b \). Now we can write the kinetic term in the Lagrangian:

\[
L_{\text{kin}} = -\frac{1}{4} G^{\mu \nu}_a G_{\mu \nu, a} - \frac{1}{4} W^{\mu \nu}_a W_{\mu \nu, a} - \frac{1}{4} B^{\mu \nu} B_{\mu \nu} \\
+ i \bar{Q}_L i \bar{\partial}_L Q_L + i \bar{u}_R i \partial_R u_R + i \bar{d}_R i \partial_R d_R + i \bar{L}_L i \bar{\partial}_L L_L + i \bar{e}_R i \partial_R e_R \\
+ (D_{\mu} \phi^*) (D^\mu \phi)
\] (1.10)

The Higgs potential term is simply

\[
L_{\phi} = -V(\phi) = -\mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2
\] (1.11)

where \( \lambda > 0 \) and \( \mu^2 < 0 \), so that spontaneous symmetry breaking occurs, as will be discussed in the next section. Finally, the Yukawa terms are

\[
L_{\text{yuk}} = -y^{u}_{ij} \bar{Q}_L u_R \tilde{\phi} - y^{d}_{ij} \bar{Q}_L d_R \phi - y^{e}_{ij} \bar{L}_L e_R \phi + \text{h.c.}
\] (1.12)

where \((\tilde{\phi})_a = i (\sigma_2 \phi^*)_a = \epsilon_{ab} \phi_b^* \) is an alternative conjugate for \( \phi \) that transforms as \((1, 2)_{-1/2}\). This “tilde-trick” allows us to write more invariants using \( \phi \) without introducing a second Higgs doublet, and relies on the fact that the \( 2^* \) of \( SU(2) \) is equivalent to the \( 2 \). These Yukawa terms will lead to mass terms for the fermions following electroweak symmetry breaking, which we now discuss.

### 1.1.2 Electroweak Symmetry Breaking

A crucial aspect of the standard model is that the \( SU(2)_L \times U(1)_Y \) symmetry of the electroweak force is spontaneously broken to the observed \( U(1)_{EM} \) at low energies. Recall the Higgs potential we wrote above,

\[
V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2
\] (1.13)

where \( \lambda > 0 \) and crucially, \( \mu^2 < 0 \). This is the famous “Mexican Hat Potential”, sketched in Fig. 1. This potential has a minimum at

\[
\langle \phi \rangle = \frac{v}{\sqrt{2}}
\] (1.14)

where we’ve defined \( v = \sqrt{-\mu^2/\lambda} = 246 \text{ GeV} \), the so-called “vacuum expectation value”, or “vev” for short. Up to an unimportant constant, we can rewrite the potential as

\[
V(\phi) = \lambda \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2.
\] (1.15)

Now, we write the complex \( SU(2)_L \) doublet \( \phi \) as

\[
\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i \phi_2 \\ \phi_3 + i \phi_4 \end{pmatrix}
\] (1.16)
where the superscripts denote the charge under $Q = T^3 + Y$, e.g.,

\begin{align*}
Q(\phi^0) &= -\frac{1}{2} + \frac{1}{2} = 0 \\
Q(\phi^+) &= \frac{1}{2} + \frac{1}{2} = 1
\end{align*}

(1.17) (1.18)

Then, $V(\phi)$ can be written

\[ V(\phi) = \frac{\mu^2}{2} \left( \sum_{i=1}^{4} \phi_i^2 \right) + \frac{\lambda}{4} \left( \sum_{i=1}^{4} \phi_i^2 \right)^2. \]

(1.19)

Note that in this form, there is a manifest $SO(4)$ global symmetry. This is an accidental symmetry of the standard model, which has some important phenomenological consequences. Without loss of generality, we choose the vev to lie along the $\phi_3$ direction: $\langle \phi_3 \rangle = v \neq 0^1$, with $\langle \phi_j \rangle = 0$ for $j = 1, 2, 4$. Put more simply,

\[ \langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \]

(1.20)

This breaks the global $SO(4)$ to $SO(3)$, or equivalently, $SU(2) \times SU(2)$ to an $SU(2)$ subgroup. The unbroken global $SU(2)$ symmetry is called custodial symmetry.

Note that a gauge transformation of $SU(2)_L \times U(1)_Y$ on $\phi$ acts as

\[ \phi \to e^{i\alpha^a T_a} e^{i\beta/2} \phi \]

(1.21)

\[ \approx (1 + i\alpha^a T_a) \left( 1 + i\frac{\beta}{2} \right) \phi \]

(1.22)

\[ \approx \left( 1 + i\alpha^a T_a + i\frac{\beta}{2} \right) \phi. \]

(1.23)

\footnote{Any other choice can be transformed into this one by an $SU(2)_L \times U(1)_Y$ gauge transformation.}
where \( T_a = \frac{1}{2} \sigma_a \). For \( \alpha^1 = \alpha^2 = 0, \alpha^3 = \beta \), the vacuum of \( \phi \) is left unchanged:

\[
\phi \rightarrow \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{i\alpha^3}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{i\beta}{2} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}
\]

(1.24)

In other words, the generator \( Q = T_3 + Y \) is the generator that leaves this choice of vacuum invariant – this is the generator of \( U(1)_{EM} \). The other generators \( (T_1, T_2, T_3 - Y) \) lead to rotations of this vacuum. The massless gauge boson corresponding to \( Q \) is the photon, while the three broken generators correspond to three massive gauge bosons, \( W^\pm \) and \( Z \). To see this in detail, consider fluctuations around the vacuum, which we can parameterize as

\[
\phi = \frac{1}{\sqrt{2}} e^{i\xi^a T'_a} \begin{pmatrix} 0 \\ v + h \end{pmatrix}
\]

(1.27)

where the \( T'_a \) are the broken generators. For global symmetries, the \( xi^i \) are Goldstone bosons. For local gauge symmetry though, we can choose unitary gauge to remove them and get the physical spectrum:

\[
\phi \rightarrow \phi' = e^{-i\xi^a T'_a} \phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}
\]

(1.28)

Now the gauge boson masses arise from the term

\[
\mathcal{L}_{\text{kin}} \supset (D_\mu \phi)^\dagger (D^\mu \phi)
\]

\[
= \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} \left( \frac{1}{2} g \sigma_b W_b^\mu + \frac{1}{2} g' B_b^\mu \right)^2 \begin{pmatrix} 0 \\ v \end{pmatrix} + \text{h-dep. terms.}
\]

(1.29)

Suppressing Lorentz indices for the moment, we have

\[
\sigma_b W_b = \sigma_1 W_1 + \sigma_2 W_2 + \sigma_3 W_3
\]

(1.30)

\[
= \sqrt{2} \begin{pmatrix} \sigma_1 + i\sigma_2 \\ \frac{W_1 - iW_2}{\sqrt{2}} \end{pmatrix} + \sqrt{2} \begin{pmatrix} \sigma_1 - i\sigma_2 \\ \frac{W_1 + iW_2}{\sqrt{2}} \end{pmatrix} + \sigma_3 W_3
\]

\[
\equiv \sqrt{2} \sigma^+ W^+ + \sqrt{2} \sigma^- W^- + \sigma^3 W^3
\]

(1.31)

where

\[
\sigma^\pm = \sigma_1 \pm i\sigma_2
\]

\[
W^\pm = \frac{1}{\sqrt{2}} (W_1 \mp iW_2)
\]

(1.32)

(1.33)

Then Eq. 1.29 becomes

\[
= \frac{1}{4} g^2 v^2 W^+ \mu W^- \mu + \frac{1}{2} \left( g^2 + g'^2 \right) \frac{v^2}{4} \left( \frac{g W_3^\mu - g' B_\mu}{\sqrt{g^2 + g'^2}} \right)^2
\]

\[
\equiv m^2_W W^\mu W^- \mu + m^2_Z Z_\mu Z^- \mu
\]

(1.34)

(1.35)

(1.36)

(1.37)
where we define
\[ Z^\mu = -g' B^\mu + g W_3^\mu \sqrt{g^2 + g'^2} = -\sin \theta_W B^\mu + \cos \theta_W W_3^\mu \tag{1.38} \]

where \( \theta_W \) is the weak mixing angle (sometimes called the Weinberg angle). From here it’s clear that a fourth field (the photon) is orthogonal to \( Z^\mu \) and remains massless:
\[ A^\mu = \frac{g B^\mu + g' W_3^\mu}{\sqrt{g^2 + g'^2}} \tag{1.39} \]

The angle \( \theta_W \) describes the rotation between the unbroken gauge eigenstates and mass eigenstates:
\[ \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix}. \tag{1.40} \]

For reference, we can write
\[ \tan \theta_W = \frac{g'}{g} \tag{1.41} \]
\[ m_W = \frac{1}{2} g v, \quad m_Z = \frac{1}{2} v \sqrt{g^2 + g'^2} = \frac{m_W}{\cos \theta_W} \tag{1.42} \]

Now we can rewrite the gauge covariant derivatives as
\[ D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} \left( W_\mu^+ T^+ + W_\mu^- T^- - i \frac{g}{\cos \theta_W} Z_\mu \left( T^3 - \sin^2 \theta_W Q \right) - ie A_\mu Q \right) \tag{1.43} \]

where the electric charge is \( e = g \sin \theta_W \). Note that all electroweak processes and interactions are described by only three independent parameters (e.g., \( e, \theta_W, \) and \( m_W \)).

1.1.3 Discrete and Approximate Symmetries of the SM

Baryon and Lepton Number

\[ C, P, CP, CPT \]

Custodial Symmetry

Find that reference on a more complete list?

1.2 Shortcomings of the SM

While the Standard Model has been remarkably successful and continues to be confirmed with astonishing accuracy at ever higher energy colliders, there are several known shortcomings. For one, the Standard model has nothing to say about gravity, and so it’s clear it cannot be valid above the Planck scale (\( M_{Pl} \sim 10^{19} \) GeV). Some of the other shortcomings include:
• Neutrino masses: in the Standard Model, the neutrinos are massless. This would appear a relatively easy problem to fix, but there are a variety of known solutions which can have different phenomenological consequences. That being said, the neutrinos do have mass, as it’s required for the flavor oscillations that have been observed. We’ll discuss these oscillations and some possible solutions to the mass problem in Section 5.

• Dark Matter and Energy: SM particles make up only 15.6% of the known matter in the universe, and only around 5% of the known energy density. The rest of the matter and energy are respectively attributed to what’s known as Dark Matter and Dark Energy. There are a number of candidates for each of these, but no conclusive evidence has been provided for any solution thus far, despite a number of experimental efforts.

• Matter/Antimatter Asymmetry: Another problem related to Cosmology is the apparent asymmetry between matter and antimatter - we have strong reason to believe that all the matter in the universe formed from a hot dense state in the early universe, but since the known asymmetries between the particles that make up the universe and their antiparticles are very small, it’s unclear why the matter didn’t simply annihilate with its antimatter counterparts. Some possible solutions of this asymmetry are known as baryogenesis and leptogenesis (referring to the sector in which the additional asymmetry arises) but neither of these scenarios has been experimentally confirmed thus far.

• The Hierarchy Problem: Another potential problem is the apparent instability of the Higgs mass to large quantum corrections. We’ll discuss this in more detail below.

There are also a number of more aesthetic issues:

• The gauge couplings of the standard model run as a function of energy and pass relatively near each other at an energy scale around $\Lambda_{\text{GUT}} \sim 10^{16}$ GeV. However, we know now that they don’t in fact intersect. Such a near intersection is often taken as a clue that there may be new physics at or just below this scale that changes the scale dependence and results in “Grand Unification”. Unfortunately, the simplest of such “Grand Unified Theories” (or GUTs), the $SU(5)$ theory, predicts a lifetime for the proton that has been experimentally ruled out.

• The standard model has 18 seemingly arbitrary parameters (the coupling strengths and particle masses), that seemingly have no underlying structure. It would be pleasing if there was a deeper theory that removed some of the “arbitrary-ness” of the theory.

• Additionally, the fermions present an enormous hierarchy of scales. For example, $m_e/m_t \sim 3 \times 10^{-6} << 1$. It’s very difficult to imagine simple phenomena that give rise to such large discrepancies, but it’s perhaps equally unappetizing to simply plug in such arbitrarily large factors by hand as well.
1.2.1 The Higgs Hierarchy Problem

The Higgs hierarchy problem deserves special notice here, as it is the crux of a lot of current
endeavours in both experiment and theory. The Higgs is the only scalar particle in the
standard model, and as discussed above, is responsible for giving mass to the weak gauge
bosons, $W^\pm$ and $Z$. In 2012, experiments at the LHC showed strong evidence for a spin 0	particle that is a remnant of the Higgs mechanism with mass $m_h \sim 125$ GeV.

The hierarchy problem is that the quantum corrections to the Higgs mass are potentially
large. These corrections arise from loops of both quarks and gauge bosons (as well as the
Higgs itself) in the scalar propagator. In a cutoff regularization scheme, we have

\[
\frac{-3\lambda^2}{8\pi^2} \Lambda^2 \approx \frac{g^2}{16\pi^2} \Lambda^2 \approx \frac{\Lambda^2}{16\pi^2} \Lambda^2
\]

Gathering all these contributions together, we find

\[
m_{h, \text{obs.}}^2 = m_{h, \text{tree}}^2 - (240 - 15 - 1) (125 \text{ GeV}) \left( \frac{\Lambda}{10 \text{ TeV}} \right)^2 
\]

Here we interpret $\Lambda$ as the scale where our description of the underlying theory is no
longer valid (i.e., it could be the scale at which new physics enters the equation). Since the
only higher energy scale we know of is the Planck scale, $M_{\text{Pl}} \sim 10^{19}$ GeV, we can see from
Eq. 1.44 that it seems very difficult to explain how the Higgs mass is only 125 GeV.

Put this way, we can view the hierarchy problem as a fine-tuning issue. If $\Lambda = 10$ TeV
above, then to get $m_{h, \text{obs.}} = 125$ GeV, we have to give the Higgs a tree mass of $m_{h, \text{tree}}^2 = 225 \times (125 \text{ GeV})^2$. This is a fine tuning of parameters at about 1 part in 100, which is a little
unsettling if we’re trying to build a theory with all $O(1)$ coefficients. Clearly, if we increase
$\Lambda$ to even higher than 10 TeV, the fine tuning problem becomes much worse. However, if we
take $\Lambda = 1$ TeV the tree level mass of the Higgs and it’s observed mass are roughly on the
same order, and there is no real fine tuning problem. This is one reason to expect new physics
at the weak scale (such as weak scale supersymmetry), and was one of the motivations for
building the LHC, beyond simply discovering the Higgs.
2 Unitarity Bounds in Particle Physics

In this section we’ll discuss constraints on scattering processes arising from unitarity. Historically, these were important in particle physics as they provided a guarantee that the LHC would find something when it started up in the mid-2000’s.

2.1 Partial-Wave Unitarity Limits

Consider a $2 \rightarrow 2$ scattering process:

\begin{equation}
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2 \equiv |f(k, \theta)|^2
\end{equation}

where $s = E_{CM}^2$, $k = \frac{1}{2} \sqrt{s - 4m^2}$ (the three-momentum of the incoming particle with mass $m$). Note that, for simplicity, we’ve assumed that both incoming particles have the same mass, $m$. $f(k, \theta)$ is the scattering amplitude familiar from ordinary quantum mechanics, and as usual, we can expand it in terms of partial waves:

\begin{equation}
f(k, \theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell(k) P_\ell(\cos \theta)
\end{equation}

where $P_\ell$ is the $\ell$-th Legendre Polynomial. We’ll show in more detail later that unitarity of the S-Matrix (or equivalently, the optical theorem) imply that

\begin{equation}
|a_\ell(k)|^2 \leq \text{Im}(a_\ell(k)) \leq 1
\end{equation}

For the elastic cross section, $\sigma(k)$, one can show that

\begin{equation}
\sigma(k) \equiv \sum_{\ell=0}^{\infty} \sigma_\ell(k) = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1)|a_\ell(k)|^2
\end{equation}

from which Eq. 2.3 implies

\begin{equation}
\sigma_\ell(k) \leq \frac{4\pi}{k^2} (2\ell + 1)
\end{equation}

This is the unitarity limit, at the level of partial waves. Now, for an “s-wave” process (i.e., only $\ell = 0$ contributes, so that $d\sigma/d\Omega$ is independent of $\theta$),

\begin{equation}
\sigma_0(k) \leq \frac{4\pi}{k^2} = \frac{16\pi}{s}
\end{equation}
where the last equality holds if \( m = 0 \). More generally, for \( m = 0 \) (or \( s \gg m \)),

\[
\sigma_\ell(s) \lesssim \frac{16\pi}{s}(2\ell + 1)
\]  

(2.7)

### 2.1.1 4-Fermi Theory

Before verifying some of the bounds above more rigorously, let’s investigate some of the consequences. An easy example is the 4-Fermi Theory. Consider \( \nu_e e^- \rightarrow \nu_e e^- \) scattering.

\[
\begin{align*}
\text{In the four-fermi theory, this scattering process comes directly from the vertex } & \mathcal{L} \supset \mathcal{G}_F \nu e \bar{\nu} e. \text{ So the cross section is} \\
\sigma(e^- \bar{\nu}_e \rightarrow e^- \bar{\nu}_e) & = \frac{G_F^2 s}{\pi} = \sigma_0.
\end{align*}
\]  

(2.8)

This is an s-wave process, so we can look only at the \( \ell = 0 \) bound. We see

\[
\frac{G_F^2 s}{\pi} < \frac{16\pi}{s}
\]  

(2.9)

which clearly is violated as \( s \rightarrow \infty \). One might object that we’ve only included the tree level process, but it’s easy to see that a loop contribution is quadratically divergent, which can only make the problem worse.

Of course, in this simple case we know this is something of a moot point, since we know the 4-Fermi interaction is only the low energy limit of the weak-force interaction shown above. But this illustrates the general principle - violations of unitarity at high energy signal the need for additional particles and interactions at a higher energy scale to save the theory. Moreover, we can get a rough estimate at what scale new physics must enter. Taking the upper limit to the inequality above, we find

\[
S^2 \lesssim \frac{16\pi^2}{G_F^4} \rightarrow \frac{\sqrt{s}}{2} = \frac{E_{CM}}{2} = \sqrt{\frac{\pi}{G_F^2}} \approx 500 \text{ GeV}
\]  

(2.10)

So we see that new physics must occur before around 500 GeV. In this case, as we mentioned, it’s the \( W \) that mediates the interaction at higher energies (\( m_W \approx 80 \) GeV), and at higher energies it’s propagator scales like \( 1/q^2 \), so \( \sigma \) doesn’t continue to grow with \( s \), and unitarity isn’t violated.
2.2 Proof of the Unitarity Limit

Here we prove the unitarity limit in the simplest case: $2 \rightarrow 2$ elastic scattering of two particles $A$ and $B$, following [3]. We start by writing the total cross section as

$$\sigma_{\text{tot}}(AB \rightarrow AB) = \frac{1}{32\pi s} \int d\cos\theta |\mathcal{M}(\theta)|^2.$$  \hspace{1cm} (2.11)

Now, we decompose the amplitude $M(\theta)$ into partial waves as follows:

$$M(\theta) = 16\pi \sum_{\ell=0}^{\infty} a_\ell (2\ell + 1) P_\ell(\cos\theta).$$  \hspace{1cm} (2.12)

Plugging this in, we find

$$\sigma_{\text{tot}} = \frac{16\pi}{s} \sum_{\ell=0}^{\infty} (2\ell + 1)|a_\ell|^2.$$  \hspace{1cm} (2.13)

Now we make use of the optical theorem, which states that

$$\text{Im}\mathcal{M}(AB \rightarrow AB, \theta = 0) = 2\sqrt{s}|p_i| \sum_X \sigma_{\text{tot}}(AB \rightarrow X)$$  \hspace{1cm} (2.14)

where the right hand side is over all possible final states. But since we’re only interested in the one particular final state $X = AB$, we have an inequality:

$$\text{Im}\mathcal{M}(AB \rightarrow AB, \theta = 0) \geq 2\sqrt{s}|p_i| \sigma_{\text{tot}}(AB \rightarrow AB).$$  \hspace{1cm} (2.15)

Plugging in our expansion in terms of the coefficients $a_\ell$, we find

$$\sum_{\ell=0}^{\infty} (2\ell + 1)\text{Im}(a_\ell) \geq \frac{2|p_i|}{\sqrt{s}} \sum_{\ell=0}^{\infty} (2\ell + 1)|a_\ell|^2.$$  \hspace{1cm} (2.16)

Note that, $|a_\ell| > \text{Im}(a_\ell)$, while the inequality above is the opposite direction – this is a serious restriction on the $a_\ell$. In fact, we could have done better by taking the incoming states $A$ and $B$ to be angular momentum eigenstates (see for example, Itzykson and Zuber [8]). In that case, all the terms in the sum are orthogonal and the inequality holds term by term. In the high energy limit, $\sqrt{s} \gg m_A, m_B$, and $|p_i| = \frac{1}{2}\sqrt{s}$, so that the above becomes

$$\text{Im}(a_\ell) \geq |a_\ell|^2.$$  \hspace{1cm} (2.17)

This inequality is satisfied by a disk in the upper half plane centered at $i/2$ and with radius 1/2. Hence, the $a_\ell$ satisfy

$$|a_\ell| \leq 1, \quad 0 \leq \text{Im}(a_\ell) \leq 1, \quad |\text{Re}(a_\ell)| \leq \frac{1}{2}.$$  \hspace{1cm} (2.18)

Equations (2.17) and (2.18) together give us the condition we used in Section 2.1.
2.3 Longitudinal Polarizations and the Goldstone Boson Equivalence Theorem

Problems with unitarity seem to re-emerge in the standard model when we consider longitudinal states of $W$-bosons as external states. Consider $e^+e^- \rightarrow W^+W^-$ scattering, taking for now the limit $m_e = 0$. This process has three diagrams at tree-level\(^2\):

Consider first the photon exchange. Each external $W$ has a polarization vector $\epsilon_\mu(|k|)$. A suitable basis for the polarization vectors (assuming the $W$ is moving in the $z$-direction) is

$$
(0, 1, 0, 0), \quad (0, 0, 1, 0), \quad \text{and} \quad \left( \frac{|k|}{m_W}, 0, 0, \frac{E_k}{m_W} \right) \tag{2.19}
$$

where $E_k = \sqrt{k^2 + m_W^2}$. These satisfy $\epsilon_\mu k^\mu = 0, \epsilon_\mu e^\mu = -1$ (where $k^\mu = (E_k, 0, 0, |k|)$. For $|k| \rightarrow \infty (s \rightarrow \infty)$,

$$
\epsilon^\mu_L \sim \frac{k^\mu}{m_W} + O\left(\frac{m_W}{E_k}\right). \tag{2.20}
$$

Now it’s clear that the part of the cross section arising from transversely polarized $W$’s is well-behaved, but the longitudinally polarized states give

$$
\frac{d\sigma}{d\cos\theta}(e^+e^- \rightarrow W^+W^-) \sim \frac{\pi \alpha^2}{4s} |\epsilon_{W^+} \cdot \epsilon_{W^-}|^2 \tag{2.21}
$$

$$
\sim \frac{\pi \alpha^2}{4s} \left( \frac{k^2}{m_W^2} \right)^2 = \frac{\pi \alpha^2}{4s} \left( \frac{s}{4m_W^2} \right)^2 \tag{2.22}
$$

$$
\sim \frac{\pi \alpha^2}{64m_W^2} s \tag{2.23}
$$

which will again violate unitarity for high $s$!

The solution here is that we haven’t included the other diagrams - once we include the diagrams from $Z$ exchange and $t$-channel neutrino exchange, this high energy behavior cancels, almost miraculously. This is actually a result of the underlying gauge symmetry - the $W$ obtains its mass from spontaneous symmetry breaking, so the three diagrams cancel.\(^2\)

\(^2\)Really, there are 4-diagrams, the extra coming from $s$-channel Higgs exchange, but the Yukawa couplings are proportional to $m_e$, which we’ve set to zero for now.
the leading behavior - this is guaranteed by the Ward identities (this was proven by ’t Hooft and Veltman, who also showed that this guarantees renormalizability).

There’s another way to see all of this that makes some of the underlying physics more clear: recall that the longitudinal polarization of \( W^\pm \) comes from the Goldstone boson of the underlying gauge symmetry. Note that, at rest, all three polarization states are indistinguishable \( (\epsilon_L = (0, 0, 0, 1)) \), but at high energies the longitudinal piece really behaves more like a charged scalar particle. We can make this substitution in any diagram, with additional terms proportional to \( m_W^2/E^2 \).

This is the “Goldstone boson equivalence theorem” – the amplitude for absorbing or emitting a longitudinally polarized \( W \) is the same as for the Goldstone boson, at leading order. So, instead of calculating \( e^+e^- \to W^+_L W^-_L \) at high energy, we can compute \( e^+e^- \to \phi^+\phi^- \):

\[
\frac{d\sigma}{d\cos \theta} (e^+e^- \to \phi^+\phi^-) = \frac{\pi\alpha^2}{4s} \sin^2 \theta \quad (2.24)
\]

which is well behaved at high energy.

There’s one last loose end to tie up: we’ve set \( m_e = 0 \) in all our calculations, but it’s easy to see that if \( m_e \) is nonzero, then \( \sigma (e^+e^- \to W^+_L W^-_L) \) receives additional terms \( \sim m_e \sqrt{s} \), which is less badly behaved but would still violate unitarity at some energy. The answer, as we anticipated in the footnote above, is that this is cancelled exactly by the s-channel Higgs exchange diagram.

2.4 Unitarity Limits and the Higgs

We can apply this unitarity bound to the Higgs boson, as well. Consider the \( H^4 \) vertex:

\[
= -6i\lambda = -6i \left( \frac{m_H^2}{2v^2} \right)
\]
and view this as $HH \rightarrow HH$ scattering to find $\sigma \propto \lambda^2 = m_H^4/4v^4$. It’s clear that for large enough $m_H$, this will numerically violate the unitarity bound\(^3\). More precisely,

$$\sigma \simeq \frac{4\pi}{64\pi^2 s} \frac{36m_H^4}{4v^4} \leq \frac{16\pi}{s}$$  \hspace{1cm} (2.25)$$

which implies $m_H \leq 4v \sim 1$ TeV.

For a more rigorous example, consider $W_L^+W_L^- \rightarrow W_L^+W_L^-$. This process has a 4 point vertex, arising from the gauge kinetic terms, an $s$-channel diagram mediated by the $Z$, $\gamma$ and Higgs, and a $t$-channel diagram. One can show that the amplitude can be written as follows:

$$\mathcal{M} = -i\sqrt{2}G_F m_H^2 \left( \frac{s}{s - m_H^2} + \frac{t}{t - m_H^2} \right)$$  \hspace{1cm} (2.26)$$

Now, if we take the limit $s, m_H^2 \gg m^2_{W,Z}$, then for $m_H \rightarrow \infty$, $\mathcal{M} \propto s$, and again we’ll violate the unitarity bound. The lesson here is that the Higgs is essential to the theory - we can’t remove it by taking it’s mass to infinity. But even for finite values of $m_H$, the limit is still important. For the $s$-wave term, with $s \gg m_H^2$,

$$\sigma_0 \sim \frac{G_F^2 m_H^4}{2\pi s} \leq \frac{16\pi}{s}$$  \hspace{1cm} (2.27)$$

demands that $m_H^2 < 4\pi \sqrt{2}/G_F \sim 1.3$ TeV. Which is a very stringent limit on the mass of the Higgs! Refining this estimate using combinations of $WW, ZZ, ZH, HH$ scattering into (possibly different) combinations of the same final states yields

$$m_H^2 \leq \frac{4\pi \sqrt{2}}{3G_F} \quad \Rightarrow \quad m_H \lesssim 700$ GeV  \hspace{1cm} (2.28)$$

This bound - sometimes called the “Lee-Quigg-Thacker bound” [9] - is an incredibly stringent bound on the mass of the Higgs, derived entirely from theory. But what exactly is this bound saying? If the standard model is weakly coupled and perturbation theory is well-defined, $m_H$ must satisfy this bound. On the other hand, the theory might be strongly coupled at some higher energy, and perturbation theory might break down (in which case, at some point we’d expect to see effects more phenomenologically similar to QCD). Alternatively, additional particles might change the computation. The above argument was done all at tree-level, but the unitarity arguments apply at all orders.

The upshot of all this is that the LHC was guaranteed to find something, as it was probing at and above these energy scales even in the first run.

### 2.5 Unitarity Limits and Dark Matter

We’ve shown how unitarity considerations were important in anticipating the mass of the Higgs boson, long before its discovery. Of course, now that a Higgs boson has been discovered with $m_H = 125$ GeV, this bound is somewhat ineffectual. However, another immediate application exists with regards to particle dark matter.

\(^3\)Of course, in this case, we already require $\lambda < 1$ for perturbation theory to be valid, but this is just a simple example.
We won’t review dark matter production in any detail here, (see e.g., [10] for a pedagogical introduction) but we’ll quickly note some of the important points. One of the most common assumptions is that dark matter was at some point in thermal equilibrium with the rest of the standard model particles in the early universe, when the density was high enough to facilitate high reaction rates for even the weakest processes. As the universe expands, the reaction rate falls off and eventually the processes “freeze-out”, leaving a relic abundance of dark matter particles, unable to find partners to annihilate with.

The matter density today is then given by

\[ \Omega \chi h^2 \approx \frac{3 \times 10^{-27} \text{cm}^3/\text{sec}}{\langle \sigma v \rangle_f} \]  

(2.29)

where \( \langle \sigma v \rangle_f \) is the velocity-averaged cross section at freeze-out. Our previous bounds for limits on the cross section from unitarity can easily be adapted to the velocity averaged case, as done in [11], for which we find

\[ \sigma \ell v \leq \frac{4\pi(2\ell + 1)}{m_\chi^2 v} \]  

(2.30)

where \( m_\chi \) is the mass of the annihilating dark matter particle. Now, one can show that the angular dependence in such a cross section always enters with a factor of \( v^2/4 \), which in these applications is a small parameter. Thus, we can safely consider only the s-wave (\( \ell = 0 \)) limit, and find

\[ \sigma v \leq \frac{4\pi}{m_\chi v} \]  

(2.31)

With this inequality, combined with Eq. 2.29, we can map the relic density today into a bound on the mass of the dark matter particles. In [11], it was required that \( \Omega \chi h^2 \lesssim 1 \), which set a limit of \( m_\chi \lesssim 340 \text{ TeV} \). Limits on the dark matter density have improved since then, though, so we can take the better limit \( \Omega \chi h^2 \lesssim 0.3 \), leading to

\[ m_\chi \lesssim 100 \text{ TeV}. \]  

(2.32)

The above method also easily gives a limit to the radius of some extended object, comprising the dark matter. If we consider an object with spin 0, and radius \( \mathcal{R}_\chi \), the highest partial wave that can contribute to the process \( \chi + \bar{\chi} \to X \) is \( J_{\text{max}} = 2m_\chi v \mathcal{R}_\chi \), so

\[ \sigma v \leq \frac{4\pi}{m_\chi^2 v} \sum_{J=0}^{J_{\text{max}}} (2J + 1) = 16\pi R_\chi^2 v. \]  

(2.33)

Requiring \( \Omega \chi h^2 \approx 0.3 \) then gives \( \mathcal{R}_\chi \gtrsim 21 \times 10^{-6} \text{ fm} \).

Unfortunately, this bound is far too high to be very helpful at current colliders, nor is it very restrictive with regards to direct detection experiments. As with the Higgs, we should also be careful interpreting what this bound actually means. Particle dark matter may have a mass higher than this limit, if it’s part of a strongly interacting theory, as discussed above.

There’s also an additional assumption here – that the dark matter was in thermal contact with the standard model in the early universe, and that this thermal freeze-out is what’s
responsible to the measured density at present. There are numerous examples of “non-thermal” dark matter candidates, and a growing number of ways to subvert this limit (cite Inflatable DM paper). That being said, we have precious few handles on the particle physics properties of dark matter, so those that we do should be taken seriously.
3 Basic Higgs Physics

We’ve already discussed some aspects of the Standard Model Higgs above - here we want to put on firmer ground exactly how the Higgs boson is produced and detected at colliders.

Look up some of the older parts of the notes with how LEP searched for the Higgs

Find references for Higgs discovery in 2011.

\[ \mathcal{L}_H \supset m_W^2 W^\mu W^\nu - \frac{1}{2} m_Z^2 Z^\mu Z^\nu \left( 1 + \frac{H}{v} \right)^2 + \frac{1}{2} (\partial_\mu H)^2 + \frac{\mu^2}{2} H^2 - \frac{\lambda v}{4} H^4 \]

\[ -m_d^i d^i_L \bar{d}^i_R \left( 1 + \frac{H}{v} \right) - m_u^i \bar{u}^i_L u^i_R \left( 1 + \frac{H}{v} \right) - m_e^i \bar{e}^i_L e^i_R \left( 1 + \frac{H}{v} \right) \]

3.1 Higgs Decay Modes

We’ll discuss how the Higgs can be produced in what comes - for now we focus on it’s detection, which must be done through decay modes.

From this, we can easily compute the Higgs decay width to fermions at tree level:

\[ \Gamma(H \rightarrow f \bar{f}) = C_f \frac{G_F m_f^2}{4\sqrt{2}\pi} \beta_f^3 m_H \]

where \( \beta_f = \sqrt{1 - 4m_f^2/m_H^2} \). Note that the decay to a top-antitop pair is disallowed for \( m_H = 125 \) GeV, as the top has a much larger mass - the top would dominate if the mass of the Higgs exceeded about 350 GeV. Thus, the decay to \( b\bar{b} \) dominates, with \( \Gamma(H \rightarrow b\bar{b}) = 2 \) MeV.

Let’s temporarily consider an arbitrary Higgs mass. There are other decay modes, e.g. to vector bosons:

\[ \Gamma(H \rightarrow VV^\dagger) = \delta_V \frac{G_F}{16\sqrt{2}\pi} \sqrt{1 - x_V} \left( 1 - x_V + \frac{3}{4} x_V^2 \right) m_H^3 \]

where \( V \) denotes either \( W \) or \( Z \), \( \delta_W = 2, \delta_Z = 1 \) (for the symmetry factor), and \( x_V = 4m_V^2/m_H^2 \). Note that this decay is proportional to \( m_H^3 \), contrary to the naive estimate one might make on dimensional grounds. This is a result of the goldstone boson equivalence theorem, since the decay to longitudinal states dominates for \( m_H \gg m_V \) (as we’re assuming for now). This resonance is extremely broad - for a very massive Higgs, we can estimate

\[ \Gamma(H \rightarrow W^+ W^- + ZZ) \sim \frac{1}{2} \left( \frac{m_H}{1 \text{ TeV}} \right)^2 m_H. \]

So for a 1 TeV Higgs boson, we’d get a width of around 500 GeV.

This exhausts the tree level decays, but it turns out that loop level processes can be very significant too. At one loop, we can have \( H \rightarrow gg \) or \( H \rightarrow \gamma\gamma \) by creating a loop of top quarks or \( W \) bosons, as shown below:
Figure 2: From [5]. The left shows a plot of the branching ratios of the SM Higgs boson as a function of the Higgs mass. The right shows the total width of the SM Higgs boson as a function of mass, compared with the width of analogues in the MSSM. See [5] for details.

Table 3: Branching ratios of various decay modes for a 125 GeV Higgs boson. The decay modes to $W^+W^-$ and $ZZ$ include off-shell decay modes, with all final states.

<table>
<thead>
<tr>
<th>final state</th>
<th>Branching Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b\bar{b}$</td>
<td>0.58</td>
</tr>
<tr>
<td>$W^+W^-$</td>
<td>0.215</td>
</tr>
<tr>
<td>$gg$</td>
<td>0.09</td>
</tr>
<tr>
<td>$\tau^+\tau^-$</td>
<td>0.06</td>
</tr>
<tr>
<td>$ZZ$</td>
<td>0.026</td>
</tr>
<tr>
<td>$\gamma\gamma$</td>
<td>0.00228</td>
</tr>
</tbody>
</table>

3.1.1 Searching for the Higgs at Colliders

For $m_H = 125$ GeV, the branching ratios are given in Table 3. The total width is $\Gamma_H \approx 4.07$ MeV.
Note that in this table we’ve included decays to $W^+W^-$ and $ZZ$, although we said before that these aren’t allowed. In the above, we’re including off-shell $W$’s and $Z$’s - one of the vector bosons must remain virtual and decay into a pair of fermions light enough to conserve energy, e.g.

![Diagram](image)

The $W^+W^-$ channel has a larger branching ratio, but these decays are harder to measure because a charged $W$ boson has to decay into either jets (which are messy) or a lepton/neutrino pair. The neutrinos are essentially non-interacting so they just show up as missing energy at the LHC. The $ZZ$ channel, on the other hand, can eventually decay into 4 leptons, and since the off-shell $Z$ has a fixed mass, the invariant mass of one pair of $\ell^+\ell^-$ must be precisely $m_Z^2$. This leads to a very strong signal that is easy to search for. Despite all this, the easiest detection channel is still decays to $\gamma\gamma$, and it was this channel that was most instrumental in the discovery of the Higgs in 2011.

The plots in Fig. 3 and 4 both show an excess around 125 GeV, however only the $\gamma\gamma$ channel was significant enough to initially claim a discovery – the signal shown in the four lepton channel would not be convincing enough without corroborating evidence from the diphoton excess.

The combined measurement from the $\sqrt{s} = 7$ and 8 TeV data is shown in Fig. 5, with the global fit converging at $m_H = 125.36$ GeV.

### 3.1.2 New Physics from Higgs Branching Ratios

Briefly, we note that we can look for new particles via the $H \rightarrow \gamma\gamma$ branching ratio. As we noted above, this decay comes primarily from a top quark loop, since the Yukawa couplings for the Higgs are proportional to the fermion mass, and $m_t \sim 173$ GeV. However, this process also gets contributions from, e.g., the supersymmetric top partner (the “stop”):

![Diagram](image)

Since there’s no reason to think the coupling of the Higgs to a stop should be particularly small, such diagrams may have significant contributions to the branching ratio. Contributions from susy also contribute to production, for example, in a gluon fusion loop:
Figure 3: From [12], the distributions of the invariant mass of diphoton candidates after all selections of the combined 7 and 8 GeV data samples. The inclusive vs. weighted samples are shown in (a) and (c) respectively, with the result of a fit to the sample with $m_H = 126.5$ GeV fixed with a background component superimposed. The residuals with respect to the fitted background are shown in (b) and (d). see [12] for more details.
Figure 4: From [12], the distribution of the four-lepton invariant mass $m_{4l}$ for events passing certain criteria, compared to the background expectation in the range 80 – 250 GeV, for the combination of $\sqrt{s} = 7$ TeV and 8 TeV data. The signal expectation for an $m_H = 125$ GeV Higgs boson is shown for guidance.
Figure 5: From [13], the observed and expected signal strengths and uncertainties for the various Higgs boson decay channels and their combination at $m_H = 125.36$ GeV. See the paper for details.
Since we don’t know the size of these couplings or the masses of the particles in the loop, we have to measure the cross-sections and branching ratios very precisely to search for new physics indirectly. The LHC hasn’t reached this level of sensitivity soon, but a future $e^+e^-$ collider (such as the ILC) would be particularly well-suited for studying these branching ratios more precisely. We’ll discuss this a little more in the next section.

Finally, we briefly note that there’s also the so-called “Higgs portal” to new physics. For example, a scalar $S$ can contribute a term

$$L_H \supset H^2 S^2 \rightarrow (\text{EWSB}) \ vHS^2.$$  

The novelty here is that this term is not suppressed by any mass scale, so its contributions could in principle be very large. Such interactions play an important role in dark matter direct detection experiments, which are starting to reach the sensitivity necessary to probe the Higgs portal today.

Going forward, the primary goals of Higgs searches will be to:

- measure the Higgs sector couplings more precisely,
- search for exotic Higgs decays, and
- search for Higgs “partners” from more complicated Higgs sectors.

### 3.2 Higgs Collider Production

Thus far, we’ve only considered the decay modes of the Higgs - of course to study these decay modes, we have to actually produce them. There are several different production channels, but they are very different depending on the type of collider being used. We’ll consider $e^+e^-$ collider production and $pp$ collider production, as these are the most commonly used today.

#### 3.2.1 Production at Electron-Positron Colliders

At $e^+e^-$ colliders, the most important production channel is $e^+e^- \rightarrow Z^* \rightarrow ZH$:

Of course, to produce a Higgs along with a $Z$ requires a center of mass energy greater than $m_H + m_Z \approx 215$ GeV. In the 90s, LEP ran at center of mass energies $\sqrt{s} \lesssim 205$ GeV, and set a limit on the Higgs mass $m_H \gtrsim 114.4$ GeV.
3.2.2 Production at Proton Colliders

Proton colliders are capable of going to much higher center of mass energies, as they don’t have to worry about the high levels of synchrotron radiation that plague electrons and positrons in circular colliders. The LHC began running in 2008 at $\sqrt{s} = 7$ TeV, and has now been upgraded to $\sqrt{s} = 13$ TeV. Future upgrades will slightly increase the center of mass energy and give substantial improvements to the already impressive luminosity.

At these high energies the protons are better described by distributions of quarks and gluons (we’ll discuss these distributions later). There are then several important processes for the production of Higgs:

Gluon Fusion, which has a cross section of about $\sigma_H \sim 44$ pb:

Vector Boson Fusion, with a cross section of around $\sigma_H \sim 3.7$ pb:

$W,Z$ associated production, with $\sigma_{HW} \sim 1.4$ pb and $\sigma_{HZ} \sim 0.87$ pb:

and $ttH$ associated production, with $\sigma_H \sim 0.5$ pb.
The gluon fusion process dominates by over a factor of 10, despite being a loop level process, because the parton distribution function of the gluon is totally dominant at low energies, which one can show in QCD. This is why it’s often more useful to think of the LHC and other pp colliders as “gluon colliders”. Note also that some of these production channels tend to result in other particles being produced, and they typically don’t involve all the constituent quarks from the proton - they therefore create very messy signals, and make the analysis of Higgs production very difficult. This is one of the reasons that an $e^+e^-$ collider, even at much lower energy, may be more useful for precision studies of the Higgs in the future.

In 2016, the LHC collected about 40 fb$^{-1}$ of data. This is a particularly convenient choice of units because we can quickly compute the number of Higgs bosons produced by multiplying the cross section. For the gluon fusion channel, we can see about $50 \times 40 \times 10^3 = 2 \times 10^6$ Higgs bosons were produced this year. Unfortunately, far fewer of these actually had data collected for them, as the triggering efficiency on gluon fusion isn’t particularly high.

Prior to the upgrade, at 8 TeV, the cross section for the Higgs was about $\sigma_H \sim 20$ pb. The initial discovery of the Higgs was announced after 5 fb$^{-1}$ of data were collected. This results in about $10^5$ total Higgs particles produced. We can also multiply by, for example, the branching ratios of the $H \rightarrow ZZ \rightarrow 4\ell$ processes, and see that about 10 events were produced in this channel.

### 3.3 Other Limits on the Higgs Mass

In addition to the bound placed by unitarity considerations, there are other limits on what the mass of the Higgs could have been. While a Higgs-like resonance has already been discovered at $m_H \simeq 125$ GeV, such considerations could be useful in extensions of the Standard Model, and as we’ll see, the measured Higgs mass implies the Standard Model is in a curiously metastable state.

#### 3.3.1 The Triviality Bound

This limit on the Higgs mass arises from considering the running of the gauge and Yukawa couplings as a function of energy. In general,

$$\frac{dg^2}{d\ln Q^2} \equiv 4\pi\beta(g^2) = b_0 g^4 + O(g^6) \quad (3.8)$$

where for the Standard Model couplings,

$$b_{g_S} = \frac{-7}{16\pi^2}, \quad b_g = \frac{1}{16\pi^2} \left( \frac{-19}{6} \right), \quad b_{g'} = \frac{1}{16\pi^2} \left( \frac{+41}{6} \right). \quad (3.9)$$

As a side note, we point out that the strong coupling $g_S$ is the only one with a negative coefficient - this implies the coupling becomes very strong at low energies, but smaller at high energies. This behavior is what’s known as “asymptotic freedom” and is responsible for much of the interesting phenomenology in QCD such as confinement and jet production. The full evolution of all standard model gauge couplings, as well as the top and bottom Yukawa couplings, is shown in Fig. 6, calculated at NNLO in [14].
Figure 6: From [14], the standard model renormalization group evolution of the gauge couplings, $g_1 = \sqrt{5/3} g'$, $g_2 = g$ and $g_3 = g_S$, as well as the top and bottom Yukawa couplings $y_t$ and $y_b$, and the Higgs quartic coupling $\lambda$. The thickness of the lines indicate the $\pm 1\sigma$ uncertainty. See the paper for more details.
For the Higgs quartic coupling, $\lambda$, we have

$$\frac{d\lambda}{d\ln Q^2} = \frac{3}{4\pi^2} \left[ \lambda^2 + \lambda y_t^2 - y_t^4 - \frac{1}{8} \lambda (g^2 + g'^2) + \frac{1}{64} (2g^4 + (g^2 + g'^2)^2) \right]$$

(3.10)

There are actually more terms, including the rest of the Yukawa couplings, but since these are proportional to the fermion masses, $y_t$ is the only one of interest. We can see that

$$y_t(v) \simeq \frac{\sqrt{2} m_t}{v} \simeq \frac{173\sqrt{2}}{246} \simeq 1$$

(3.11)

while even $y_b$ would be over a factor of 40 lower.

Now, for masses $m_H \gtrsim 350$ GeV, $\lambda(v^2) \gtrsim 1$, so we can neglect $y_t$ and the gauge couplings and write

$$\frac{d\lambda}{d\ln Q^2} = \frac{3}{4\pi^2} \lambda^2 \quad \Rightarrow \quad \frac{d\lambda}{\lambda^2} = \frac{3}{4\pi^2} d\log Q^2$$

(3.12)

or,

$$\frac{1}{\lambda(v^2)} - \frac{1}{\lambda(Q^2)} = \frac{3}{4\pi^2} \ln Q^2 - \frac{3}{4\pi^2} \ln v^2$$

$$\Rightarrow \quad \lambda(Q^2) = \frac{\lambda(v^2)}{1 - \frac{3\lambda(v^2)}{4\pi^2} \ln \left( \frac{Q^2}{v^2} \right)}.$$ 

(3.13)

This diverges at the so-called (in analogy with QED) Landau Pole:

$$Q_{LP} = v \exp \left( \frac{2\pi^2}{3\lambda(v^2)} \right).$$

(3.14)

For a pure $\lambda H^4$ theory, this theory makes sense for $Q \to \infty$ only if $\lambda(v^2) = 0$, i.e., the theory is trivial. However, all we really need is that $\lambda$ be finite up to some energy scale $\Lambda < Q_{LP}$. Rearranging the above, we find

$$\lambda(v^2) < \frac{2\pi^2}{3\ln \left( \frac{\Lambda v}{v} \right)}$$

(3.15)

or, since $m_H^2 = 2\lambda v^2 = \lambda \sqrt{2}/G_F$,

$$m_H < \sqrt{\frac{2\sqrt{2}\pi^2}{3G_F \ln \left( \frac{\Lambda}{v} \right)}}.$$ 

(3.16)

This is known as the “triviality bound” on the Higgs mass. Unfortunately, it requires inputting a new scale $\Lambda$ to make a prediction. For $\Lambda \sim m_{Pl} \sim 10^{19}$ GeV, for example, we find $m_H \lesssim 140$ GeV. On the other hand, for $\Lambda \sim 1.5$ TeV, $m_H \lesssim 650$ GeV, which is far less stringent. This would have been particularly meaningful if the Higgs mass had a much larger value, as it would set an upper limit on the scale of new physics. Unfortunately, if we invert this bound and plug in $m_H \sim 125$ GeV, we find a relatively meaningless upper bound on $\Lambda$ that is far above the Planck scale.

A more rigorous calculation than our simplified version above yields the limits illustrated in Fig. 7.
3.3.2 The Stability Bound

There is also a lower bound on $m_H$ – if $\lambda$ is small, $m_H$ is small, and the $y_t^4$ term is the most important. The leading order equation is then

$$\frac{d\lambda(Q^2)}{d\ln(Q^2)} = -\frac{3y_t^2}{4\pi^2}. \tag{3.17}$$

For constant $y_t^4$, we find

$$\lambda(Q^2) = \lambda(v^2) - \frac{3y_t^4}{4\pi^2} \ln \left( \frac{Q^2}{v^2} \right) \tag{3.18}$$

which becomes negative for

$$\lambda(v^2) < \frac{3y_t^4}{m_H^2} \ln \left( \frac{Q^2}{v^2} \right), \tag{3.19}$$

The full renormalization group evolution of $\lambda$ is shown in Fig. 8.

Now recall the general shape of the Higgs potential. For a stable vacuum to exist, we need $\lambda(\Lambda^2) > 0$, or,

$$m_H > \sqrt{\frac{3y_t^4}{\sqrt{2\pi G_F}}} \ln \left( \frac{Q}{v} \right). \tag{3.20}$$

For $\Lambda = 1.5$ TeV, this implies $m_H > 85$ GeV. However, the full phase space depends strongly on both $m_H$ and $m_t$. The relevant parameter space is illustrated in Fig. 9, borrowed

\footnote{Of course, $y_t$ also runs with scale, but we can take it as constant to first approximation here.}
Figure 8: From [14]. The renormalization group evolution of $\lambda$, varying the values of $m_t$, $m_H$ and $\alpha_S$ by $\pm 3\sigma$. See the text for details.

from [16], where the different regions are labelled based on whether the Higgs vacuum is stable, unstable, or “meta-stable”. Curiously, the current measurements seem to indicate that our universe lies in a metastable vacuum, although the stable region is not yet conclusively ruled out.
Figure 9: From [16], the phase diagram for the Standard Model in terms of the Higgs and top pole masses. The left diagram shows the full parameter space, while the right shows a zoom in of the region most relevant to the current experimental measurements. See the paper for details.

4 CP Violation and Flavor Physics

In this section we’ll discuss a variety of distinct but related topics dealing with the Yukawa sector of the standard model.

4.1 Yukawa Interactions

4.2 The CKM Matrix

4.2.1 The GIM Mechanism

4.3 CP Violation

4.3.1 Kaon Systems
5 Neutrino Physics

In the following sections we’ll discuss neutrinos. Neutrinos are massless in the Standard Model, but as we’ll see, much of their interesting phenomenology requires them to have masses. Therefore, we must explicitly add beyond the standard model (BSM) physics. Fortunately, a great deal of these properties can be explored without paying too much attention to the underlying model in a way that will be made clearer in the next section.

5.1 Neutrino Mass Terms

As mentioned above, there is no mass term for neutrinos that can be added directly to the standard model. A Dirac mass term would require an additional right handed fermion field $N_R$ to complement the $E_R$ – without this, no term $\propto m_D \bar{\nu}_L N_R$ exists. In addition, a Majorana mass term ($\propto \frac{1}{2} m_M (\nu^T_L C \nu_L + h.c.)$) would violate lepton number conservation, and is therefore explicitly forbidden in the ordinary Standard Model.\(^5\)

In all the following sections, we’ll distinguish between active and sterile neutrinos. Active neutrinos are part of the $SU(2)_L$ doublet, and therefore participate in the weak interactions. Sterile neutrinos on the other hand are singlets under $SU(2)_L$, and therefore do not couple directly to the weak force. However, sterile neutrinos can still mix with active neutrinos (as we’ll discuss below), and are therefore still very important phenomenologically.

Before discussing how we can generate different mass terms for neutrinos, we review some notation below.

- $\psi_L \equiv P_L \psi \equiv \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$,
- $\bar{\psi}_L \equiv \psi_L^\dagger \gamma^0 \equiv (P_L \psi)^\dagger \gamma^0$,
- doublets: $L_L \equiv \begin{pmatrix} \nu_{eL} \\ \bar{e}_{L} \end{pmatrix}$,
- $\psi^C_R \equiv C \bar{\psi}^T_L$, ($C = -i\gamma^2\gamma^0$ in chiral rep.).

With this notation, the $CP$ conjugates work as follows:

$$\text{sterile RH } \nu : \quad \nu_R \overset{CP}{\leftrightarrow} \nu^C_L$$
$$\text{active LH } \nu : \quad \nu_L \overset{CP}{\leftrightarrow} \nu^C_R. \quad (5.1)$$

With these in hand, we can discuss different ways of generating Dirac or Majorana mass terms.

\(^5\)Lepton number ($L$) and Baryon number ($B$) are anomalous in the Standard Model (they are violated by quantum corrections), but the combination $B - L$ is not anomalous, and is therefore a good symmetry. A Majorana mass term would violate $B - L$, and therefore new physics is still needed to generate a Majorana mass for neutrinos.
5.1.1 Dirac Masses

A Dirac neutrino mass term has the form
\[ \mathcal{L} = -m_D (\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L) \equiv -m_D \bar{\nu}_D \nu_D \] (5.3)

where we’ve defined \( \nu_D = \nu_L + \nu_R \). Note that this requires a Dirac neutrino, and changes \( SU(2)_L \) isospin \( \delta t^3 = 1/2 \). Provided the Dirac neutrino exists, this mass can be generated via the Higgs mechanism by adding a tiny Yukawa coupling:
\[ y_\nu \bar{L}_L \phi \nu_R. \] (5.4)

The Yukawa coupling must be tiny because the neutrino masses are much smaller than the other leptons. We don’t know the precise neutrino masses today, but a combination of Cosmological input and Oscillation data sets bounds on the sum of the neutrino masses:
\[ 0.06 \text{ eV} \leq \sum m_\nu \lesssim 0.3 \text{ eV} \] (5.5)

(note that these limits are independent of whether the masses are Majorana or Dirac). Running through the usual Higgs mechanism, we find
\[ m_\nu \sim \frac{y_\nu v}{\sqrt{2}} \] (5.6)

which implies that \( y_\nu \sim \sqrt{2} m_\nu/v \approx 0.1 \text{ eV}/100 \text{ GeV} \approx 10^{-12} \). This can be compared to the other leptons: \( y_t \sim 1 \) and \( y_e \sim 10^{-5} \). This is a tremendous hierarchy, which is unappealing but not inconsistent.

Note that for Dirac masses, there is a conserved lepton number:
\[ \nu_{L,R} \to e^{-i\alpha} \nu_{L,R}, \] (5.7)
due to a global \( U(1) \) symmetry.

5.1.2 Majorana Masses

Active Neutrinos

For an active neutrino to have a Majorana mass, we want a term of the form
\[ \mathcal{L} = -\frac{m_M}{2} \left( \bar{\nu}_L \nu^C_R + \bar{\nu}^C_R \nu_L \right) \] (5.8)

\[ = -\frac{m_M}{2} \left( \bar{\nu}_L C \nu^T_L + \nu^T_L C \nu_L \right) \] (5.9)

Note that this violates lepton number by 2 units. A term of this form also has other important phenomenological consequences, as it can mediate what’s known as “neutrinoless double beta decay”, or \( 0\nu\beta\beta \). This process can occur in some elements where the ordinary single beta decay of the nucleus \( (n \to p + e^- + \bar{\nu}_e) \) is energetically forbidden, so that the dominant decay mode of the nucleus is to simultaneously have two neutrons decay into protons + electrons and neutrinos. The ordinary two neutrino double beta decay has been observed in a few different elements (including somewhat recently, Xe\(^{136}\)), but the neutrinoless mode has yet to be detected.
One way of understanding the neutrinoless mode is to note that, if the neutrinos are Majorana fermions, they are “their own antiparticle”. Thus, we can imagine the two final state neutrinos instead annihilating into each other as an intermediate process in the diagram, leaving only the two final state electrons (and protons). Alternatively, one can draw the diagram and consider an additional Feynman rule - the addition of a “Majorana mass insertion” (see below). Several experiments are already underway searching for neutrinoless double beta decay, as it would provide evidence that the neutrino mass arises (at least in part) from a Majorana term.

Note also that these Majorana mass terms violate $SU(2)_L$ isospin by $\Delta t^3 = 1$. This restricts the ways it could be generated. One way would involve a Higgs triplet, which we’ll explore in detail below. Alternatively, one can add a dimension 5 operator

$$\sim \frac{\kappa}{\Lambda} (HHL_LL_L)$$ (5.10)

Once the Higgs acquires a vev,

$$m_M \sim \frac{\kappa}{\Lambda} v^2 \sim 0.1 \text{ eV}$$ (5.11)

where the last approximation is based on the actual neutrino masses. Thus, for $\kappa$ to be $\mathcal{O}(1)$, we need the scale $\Lambda \sim 10^{14}$ GeV. This is tantalizingly close to the GUT scale, which has inspired countless GUT related models for the neutrino masses.

**Sterile Neutrinos** For sterile neutrinos, a Majorana mass takes the form

$$\mathcal{L} \ni -\frac{m_S}{2} \left( \bar{\nu}_L^C \nu_R + \bar{\nu}_R^C \nu_L^C \right)$$ (5.12)

$$= -\frac{m_S}{2} \left( \bar{\nu}_L^C \nu_L^C T + \nu_L^C C \nu_L^C \right).$$ (5.13)

A term of this form does not change $SU(2)_L$ isospin ($\Delta t^3 = 0$) – as it is for sterile neutrinos – and thus could be generated with a Higgs singlet.
5.1.3 Some Comments on Phases

Consider the Lepton Yukawa terms in the Lagrangian, along with a new Majorana mass for the neutrinos.

\[
\mathcal{L} \supset -y_{ij}^L L_i \phi e_{Rj} + \frac{\kappa_{ij}}{\Lambda} \phi \phi L_i L_{Lj} + \text{h.c.}
\] (5.14)

Similar to the analysis for the CKM Matrix, we can write

\[
y_L = U_L^{\dagger} D_L U_L
\]

where \(U_L, W\) are unitary and \(D\) is diagonal. Then, we can rotate the fields

\[
e_{Li} \rightarrow U_{Li} e_{Lj}
\] (5.16)

\[
e_{Ri} \rightarrow W_{Li} e_{Rj}
\] (5.17)

Then, as usual we take \(\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \phi^0 \end{pmatrix}\), so that

\[
\mathcal{L}_{\text{Yuk}} \supset -\frac{1}{\sqrt{2}} \phi^0 \bar{\epsilon}_L U_L \left( U_L D_L U_L^{\dagger} \right) W_L e_R
\] (5.18)

\[
= -\frac{1}{\sqrt{2}} \phi^0 \bar{\epsilon}_L U_L \left( U_L D_L W_L^{\dagger} \right) W_L e_R
\] (5.19)

\[
= -\frac{1}{\sqrt{2}} \phi^0 \bar{\epsilon}_L D_L
\] (5.20)

And we see that we’ve generated Dirac masses for the charged leptons. But now we have to consider the weak interactions involving \(W\) bosons. For example, the charged current contains

\[
J_W^- = \frac{1}{\sqrt{2}} \bar{e}_L \gamma^\mu \nu_L
\] (5.21)

\[
\rightarrow \frac{1}{\sqrt{2}} \bar{e}_L U_L^{\dagger} \gamma^\mu \nu_{Li}
\] (5.22)

In the SM (with massless neutrinos), we can rotate \(\nu_L \rightarrow U_L \nu_L\) freely. However, now we have Majorana mass terms, so we cannot simply rotate the left handed neutrino fields and preserve the real masses. The result is that we wind up with an extra \(3 \times 3\) matrix, schematically:

\[
\begin{pmatrix} e & \mu & \tau \end{pmatrix} (3 \times 3) \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}
\] (5.23)

where the \(3 \times 3\) matrix corresponds to \(U_L^{\dagger}\), in this particular term. Now one may ask how many angles and phases are necessary to describe \(U_L^{\dagger}\)? We know that \(U_L\) is unitary, so there are at least initially \(3^2 = 9\) degrees of freedom. If \(U_L^{\dagger}\) were real, we’d have \(\frac{3}{2}(3 - 1) = 3\) degrees of freedom (three real angles), so we know that the 9 degrees of freedom consist of 3 angles and 6 phases.

But not all of these 6 phases are physical. We can rotate away 3 of the phases via

\[
e_{Li} \rightarrow e^{i\alpha_i} e_{L_i}, \quad i = 1, 2, 3 \ (e, \mu, \tau).
\] (5.24)
But we cannot rotate the $\nu_L$'s, since the Majorana masses would become:

$$\frac{\kappa u^2}{\Lambda} \nu_L \nu_L \rightarrow \frac{\kappa u^2}{\Lambda} e^{2i\beta} \nu_L \nu_L$$

(5.25)

which is manifestly not invariant, and not real! So, we end up with 3 physical angles, and 3 physical phases. Note though, that this is 2 phases more than were in the CKM matrix (which had only one physical phase, $\delta_{CP}$). The two additional phases are called “Majorana phases”, as they arise because of the Majorana mass terms. They can appear in processes such as neutrinoless double beta decay, where the Majorana mass insertion plays an important role.

There are of course, a number of parameterizations of the matrices $U_L$. A popular one is to write

$$U_L^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & \sin \theta_{23} \\ 0 & -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix} \begin{pmatrix} \cos \theta_{13} & 0 & e^{-i\delta} \sin \theta_{13} \\ 0 & 1 & 0 \\ -e^{+i\delta} \sin \theta_{13} & 0 & \cos \theta_{13} \end{pmatrix} \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

(5.26)

When parameterized in this form, the neutrino mixing matrix is known as the PMNS matrix, after the authors Pontecarro, Maki, Nakagawa, and Sakata. Usually though, the PMNS matrix refers to the Dirac case, where the two Majorana phases $\alpha_1$ and $\alpha_2$ can be safely absorbed by rotating the neutrino fields.

The three angles in the PMNS matrix, $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ have all been measured – $\theta_{13}$ only recently. We’ll discuss how they are measured in subsequent sections, but here we should note that $\theta_{13}$ was found to be about 9°, which is small but not too close to zero. As we’ll describe in more detail later, if $\theta_{13}$ had been much closer to zero, a measurement of $\delta_{CP}$ in the neutrino sector would have been impossible, as the combination would always vanish when arising through oscillations.

5.1.4 The Seesaw Mechanism

In this section we’ll describe the generic *seesaw mechanism*, one of the most popular ways of generating small neutrino masses. A number of variations have been described in the literature, but the basic idea remains the same. For simplicity, we consider only one generation of neutrinos – the calculation for three generations is nearly identical.

The idea is that if both active and sterile neutrinos exist, we can write down both Majorana and Dirac mass terms. Regardless of the origin of the masses, the Lagrangian takes the form

$$\mathcal{L} \supset -\frac{1}{2} \left( \bar{\nu}_L \bar{\nu}_L^C \right) \begin{pmatrix} m_T & m_D & m_S \end{pmatrix} \begin{pmatrix} \nu_L^C \\ \nu_R \end{pmatrix} + \text{h.c.}$$

(5.28)

where all the $m$’s are masses. We can imagine $m_D$ arising from Yukawa terms, $m_S$ from a singlet Higgs, and $m_T$ from a triplet. In general, there is nothing preventing a theory from containing all three.

Now we assume that $m_D$ is at approximately the weak scale, $v = 246$ GeV (i.e., it arises from $\mathcal{O}(1)$ Yukawa couplings). Next, assume that $m_S$ is at some higher scale, such as the...
GUT scale $\Lambda \sim 10^{15}$ GeV and that $m_T$ is negligibly small or zero (perhaps there is no triplet Higgs). Then we can diagonalize the matrix to find two mass eigenstates

$$\nu_{1L} \simeq \nu_L, \hspace{1cm} m_1 \sim O\left(\frac{m_D^2}{m_S}\right)$$

$$\nu_{2L} \simeq \nu^C_L, \hspace{1cm} m_2 \sim O\left(m_S\right)$$

where $\nu_{1L}$ is active and $\nu_{2L}$ is sterile. If $m_S \gg m_D$ as we’ve assumed, then $\nu_L$ is naturally very light and $m_1 \sim \left(\frac{m_D}{m_S}\right) m_D \ll \text{weak scale}$. To obtain masses in the 0.1 eV range, for example, we need $m_S \sim 10^{14}$ GeV. We see that the “seesaw” effect of introducing a very large scale can indeed make one neutrino mass eigenstate very light, as desired.

### 5.2 Vacuum Oscillations

Now we wish to describe neutrino oscillations. As we’ll see, these are very similar to the oscillations of $K$-mesons discussed before. The basic idea is that in experiments we produce interaction eigenstates,

$$\nu_e, \hspace{1cm} \nu_\mu, \hspace{1cm} \nu_\tau.$$ 

On the other hand, it’s the eigenstates of the Hamiltonian which describe propagation in spacetime – the mass eigenstates:

$$\nu_1, \hspace{1cm} \nu_2, \hspace{1cm} \nu_3.$$ 

In general, we can write

$$|\nu_\alpha\rangle = U^*_{\alpha i} |\nu_i\rangle$$

where $i = 1, 2, 3$ and $\alpha = e, \mu, \tau$ and $U_{\alpha i}$ a unitary matrix. For simplicity, let’s consider the case of two flavors:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

where $\theta$ is the one mixing angle necessary for the two flavor case. Now, imagine that at $t = 0$, we have a pure muon neutrino sample,

$$|\nu(t = 0)\rangle = |\nu_\mu\rangle$$

with momentum $p$ (this could be produced, e.g., via a beam of pions decaying: $\pi^+ \rightarrow \mu^+ + \nu_\mu$). We want to compute the probability for detecting electron neutrinos some time later (The electron neutrinos could be detected via the reaction $\nu_e + n \rightarrow p + e^-$, where the charged products are easily tracked and measured.)

We have,

$$|\nu(0)\rangle = \nu_\mu = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle$$

so, as in ordinary quantum mechanics,

$$|\nu(t)\rangle = -\sin \theta |\nu_1\rangle e^{-iE_1 t} + \cos \theta |\nu_2\rangle e^{-iE_2 t}$$

where

$$E_i = \sqrt{p^2 + m_i^2} \approx E + \frac{m_i^2}{2E}$$
and we can assume $E \approx |p|$ (since the neutrino masses are small). Then,

$$|\nu(t)\rangle = \left(-|\nu_1\rangle \sin \theta e^{-i \frac{m_1^2 t}{2E}} + |\nu_2\rangle \cos \theta e^{-i \frac{m_2^2 t}{2E}} \right) e^{-iEt} \quad (5.37)$$

Now, after travelling a distance $L = t$, we can compute

$$P_{\nu_\mu \rightarrow \nu_e}(t) = |\langle \nu_e | \nu_\mu(t) \rangle|^2 = \sin^2 \theta \cos^2 \theta \left| e^{-i \frac{m_1^2 L}{2E}} + e^{-i \frac{m_2^2 L}{2E}} \right|^2 = \frac{1}{4} \sin^2 2\theta \left( 1 - e^{-i \frac{L}{2E} \Delta m^2} \right)^2$$

$$= \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2 L}{4E} \right) \quad (5.39)$$

where $\Delta m^2 = m_2^2 - m_1^2$. To get a rough feel, we can plug in some numbers:

$$P_{\nu_e \rightarrow \nu_\mu}(t) = \sin^2 2\theta \sin^2 \left( \frac{1.27 \Delta m^2 (eV^2) L (km)}{4E (GeV)} \right). \quad (5.40)$$

Note that $P$ is independent of the sign of $\Delta m^2$ – we can’t tell which of the masses is greater (much more on this later). Experimentally we can control both $L$ and $E$ in the lab (or by design of the experiment), while nature controls both $\Delta m^2$ and $\theta$. Roughly, we can see that

$$\frac{\Delta m^2 L}{E} \begin{cases} \ll 1 \Rightarrow P \approx 0 \\ \sim 1 \Rightarrow P \sim \mathcal{O}(1) \\ \gg 1 \Rightarrow P \sim \frac{1}{2} \sin^2 2\theta \end{cases} \quad (5.43)$$

In the first case, the probability is negligible and measurements of both $\theta$ and $\Delta m^2$ are impossible. In the final case, the oscillations average out so that the effects of $\Delta m^2$ are hidden but $\theta$ is still measurable. Only in the middle case are we sensitive to both $\theta$ and $\Delta m^2$. From this we can also see that to observe oscillations with very small $\Delta m^2$ we require a very large ratio $L/E$. We’ll discuss several explicit examples of this in later sections.

In the more general case, we have

$$P_{\alpha \beta} \equiv P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha \beta} - 4 \sum_{j>i} \text{Re} \left( U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \right) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{4E} \right) + 2 \sum_{j>i} \text{Im} \left( U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \right) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{2E} \right) \quad (5.44)$$

Finally, note that while $\delta_{CP}$ hasn’t appeared in the above, oscillations are sensitive to $CP$ violation. In this case, $CP$ violation would manifest as an inequivalence of $P_{\alpha \beta}$ and $P_{\bar{\alpha} \bar{\beta}}$, which, assuming $CPT$ invariance would imply $P_{\alpha \beta} \neq P_{\beta \alpha}$. On the other hand, the two Majorana phases make no difference - they can’t be measured in any hypothetical oscillation experiment.
5.3 The MSW Effect

Before describing neutrino oscillation measurements, we’ll provide a sketch of the so-called MSW effect, for Mikheyev, Smirnov, and Wolfenstein. The MSW effect describes how neutrino oscillations change as the neutrinos propagate through a matter background (e.g., through the earth or from the center of the sun).

The basic idea behind the MSW effect is that while the neutral current interactions, e.g., are the same for all $\alpha$, the charged current interactions between neutrinos and electrons in matter are different for each flavor. In particular, $\nu_e$ is affected differently than $\nu_{\mu}$ and $\nu_{\tau}$, so the neutrinos have a different effective mass in matter.

As we did for oscillations, we’ll consider here the two neutrino case to avoid cumbersome notation – one can view this to good approximation as taking $\nu_e$ along with a well chosen linear combination of $\nu_{\mu}$ and $\nu_{\tau}$, which we’ll denote $\nu_\alpha$. In the mass basis, the vacuum Hamiltonian for $\nu_1, \nu_2$ is

$$H = p^I + \left( \begin{array}{cc} m_1^2 / 2E & 0 \\ 0 & m_2^2 / 2E \end{array} \right)$$

and the neutrinos satisfy the Schrödinger equation:

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = H \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \quad (5.46)$$

In the interaction basis,

$$\begin{pmatrix} \nu_e \\ \nu_\alpha \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \quad (5.47)$$

We denote the mixing matrix $U$ and write

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = U^T \begin{pmatrix} \nu_e \\ \nu_\alpha \end{pmatrix} \quad (5.48)$$

So the Schrödinger equation becomes

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\alpha \end{pmatrix} = U H U^T \begin{pmatrix} \nu_e \\ \nu_\alpha \end{pmatrix} \quad (5.49)$$

and we now have an interacting Hamiltonian,

$$H_{\text{int}} \approx \left( p + \frac{m_1^2 + m_2^2}{4E} \right) \mathbb{I} + \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \quad (5.50)$$

Thus far, we’ve discussed only the Hamiltonian in vacuum, i.e., without the matter effects we wish to discuss. Without being too specific, we can add an extra potential due to matter, that is flavor diagonal:

$$H_{\text{int}} \rightarrow H_{\text{int}} + \begin{pmatrix} V_e & 0 \\ 0 & V_\alpha \end{pmatrix} \quad (5.51)$$

$$= H_{\text{int}} + \begin{pmatrix} V_e & 0 \\ 0 & V_\alpha \end{pmatrix} + \begin{pmatrix} V_e - V_\alpha & 0 \\ 0 & 0 \end{pmatrix} \quad (5.52)$$
where in the last line we’ve separated out the piece not proportional to the identity. So, in matter:

\[ H_{\text{int}} \simeq \left( p + V_\alpha + \frac{m_1^2 + m_2^2}{4E} \right) \mathbb{1} + \left( V_e - V_\alpha - \frac{\Delta m^2}{4E} \cos 2\theta \quad \frac{\Delta m^2}{4E} \sin 2\theta \right) \]  

(5.53)

Now, it can be shown that

\[ V_e - V_\alpha \simeq \sqrt{2} G_F n_e \]  

(5.54)

where \( n_e \) is the number density of electrons in the matter the neutrinos are propagating through. Importantly, note that the sign of the above equation is the opposite for antineutrinos. Now, ignoring the pieces proportional to the identity,

\[ H_{\text{matter}} \simeq \left( \frac{1}{\sqrt{2}} G_F n_e - \frac{\Delta m^2}{4E} \cos 2\theta \quad \frac{\Delta m^2}{4E} \sin 2\theta \right) \left( \begin{array}{cc} \Delta m^2 & \sin 2\theta + A \\ \sin 2\theta & \cos 2\theta - A \end{array} \right) \]  

(5.55)

(5.56)

where for simplicity we’ve defined

\[ A = \frac{2\sqrt{2} G_F n_e E}{\Delta m^2}. \]  

(5.57)

Now, we can diagonalize to get the new mass eigenstates in matter. Write

\[ H_m = \frac{\Delta m^2}{4E} \left( \begin{array}{cc} -\cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & \cos 2\theta_m \end{array} \right) \]  

(5.58)

so that this looks just like our old oscillation probability, with \( \theta \to \theta_m \) and \( \Delta m^2 \to \Delta m^2_m \):

\[ P_{\nu_e \to \nu_\alpha} = \sin^2 2\theta_m \sin^2 \left( \frac{\Delta m^2_m L}{4E} \right) \]  

(5.59)

and

\[ \left( \begin{array}{c} \nu_e \\ \nu_\alpha \end{array} \right) = \left( \begin{array}{cc} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{array} \right) \left( \begin{array}{c} \nu_{1m} \\ \nu_{2m} \end{array} \right). \]  

(5.60)

A bit of algebra reveals

\[ \Delta m^2_m = C \Delta m^2 \]  

(5.61)

\[ \sin 2\theta_m = \frac{1}{C} \sin 2\theta \]  

(5.62)

where \( C = \sqrt{(\cos 2\theta - A)^2 + \sin^2 2\theta} \). Or,

\[ \tan 2\theta_m = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2} G_F n_e E}. \]  

(5.63)

Note that in principle, matter effects are sensitive to not only \( \Delta m^2 \), but also its sign.

Also note that there is a resonance condition: if \( A = \cos 2\theta \), \( \tan 2\theta_m \to \infty \) and the probability \( P(\nu_e \to \nu_\alpha) \) becomes maximal. This has important consequences for the solar neutrino problem, as we’ll discuss in the next section.
5.4 Neutrino Measurements

In the following sections we’ll detail some of the original neutrino problems that are solved by either oscillations or matter effects, and show how these allowed a measurement of the neutrino oscillation parameters. We’ll then give a brief overview of the current known oscillation parameters and the outstanding problems with the neutrino mass hierarchy. In many of these computations, there are substantial complications that we’re ignoring, but we hope to give the general idea.

5.4.1 Atmospheric Neutrinos

When cosmic ray protons interact with particles in the atmosphere, typically many charged pions are produced. These produce neutrinos via the reactions

\[
\pi^- \rightarrow \mu^- + \bar{\nu}_\mu \quad \text{(5.64)}
\]

\[
\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \quad \text{(5.65)}
\]

Neglecting the difference between particles and antiparticles (as the opposites are produced in \(\pi^+\) decays, anyway), we can easily count that two muon neutrinos are produced for every electron neutrino. However, the observed ratio is nearly one to one. Since \(\theta_{13}\) is relatively small, we can neglect \(\nu_e \rightarrow \nu_\tau\) oscillations as the culprit, and thus we expect that it must be \(\nu_\mu \rightarrow \nu_\tau\) oscillations responsible for the discrepancy. This allows a measurement of \(\theta_{23}\), which is found to be near maximal: \(\theta_{\text{atm}} = \theta_{23} \approx 45^\circ\). It also allows a measurement of the mass splitting, \(\Delta m^2_{23} \approx 3 \times 10^{-3} \text{ eV}^2\).

5.4.2 Solar Neutrinos

One of the oldest problems in neutrino physics is the so-called “Solar neutrino problem”. Without going into too much detail on stellar physics, we can consider the reactions in the sun, such as

\[
4p \rightarrow \cdots \rightarrow ^4He + 2e^+ + 2\nu_e. \quad \text{(5.66)}
\]

Nuclear reactions within the sun are well understood. Importantly, all of the reactions produce only electron neutrinos in their final state. One can compute the flux of the electron neutrinos produced in the sun and compare the expected number reaching earth to the number observed – this was done as early as 1968. However, the number of electron neutrinos observed on earth is far too small.

It’s commonly stated that the discrepancy is due to oscillations, but the story isn’t quite that simple. Assume the electron neutrinos produced in the sun travel outwards, so that \(n_e\) decreases as they move outwards. For resonance, we need \(A = \cos 2\theta\), or

\[
E_{\text{res}} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F n_e}. \quad \text{(5.67)}
\]

Note that \(E_{\text{res}}\) increases as \(n_e\) decreases. Therefore, any \(\nu_e\) produced in the core will only encounter a resonance if their initial energy is greater than the minimum energy for resonance:

\[
E_\nu > E_{\text{min}} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F n_{e,\text{core}}}. \quad \text{(5.68)}
\]
For our sun, \(n_{e,\text{core}} \simeq 3 \times 10^{31} \text{ m}^{-3}\). If we take \(\Delta m^2 \sim 8 \times 10^{-5} \text{ eV}^2\), \(\theta \simeq 30^\circ\), we find \(E_{\text{min}} \sim \mathcal{O}(\text{MeV})\).

Near the Sun’s center, \(n_e\) is huge, so \(\sin 2\theta_m \approx 0\) and \(\cos 2\theta_m \approx -1\), meaning \(\theta_m \simeq \pi/2\). Therefore,
\[
\begin{pmatrix}
\nu_e \\
\nu_\alpha
\end{pmatrix}
\simeq
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\nu_{1,m} \\
\nu_{2,m}
\end{pmatrix}
\tag{5.69}
\]
and the flavor eigenstate \(\nu_e \simeq \nu_{2,m}\). So, while ordinarily the interaction and mass eigenstates are different, here the electron neutrinos being produced in nuclear reactions are essentially produced as a pure sample of mass eigenstate \(\nu_{2,m}\) for the effective mass matrix. Assuming the density changes slowly enough, as the neutrinos propagate outward they remain in eigenstate \(\nu_{2,m}\). However, once they emerge from the sun and enter the vacuum, the Hamiltonian is no longer shifted but they are already in the eigenstate \(\nu_2\).

At this point, \(\theta_m = \theta\), and no oscillations take place as the \(\nu_2\) travel to earth (as they are already in an eigenstate of the Hamiltonian). Now, the probability of observing them as electron neutrinos on earth some distance away is independent of their energy, distance travelled, or \(\Delta m^2\): since they are simply in a different basis, we’ll find
\[
P_{\nu_e \rightarrow \nu_e} = \sin^2 \theta_{12} \simeq 0.31
\tag{5.70}
\]
allowing a measurement of \(\theta_{12}\) based on solar neutrinos.

However, this was only for electrons in the core produced with energies greater than the minimum energy for resonance. For those with \(E_\nu = E_{\nu,\text{min}}\) or produced in a region where there is a smaller electron density, \(\nu_e\) will be an admixture of \(\nu_1\) and \(\nu_2\), oscillations will take place, and we’ll find
\[
P_{\nu_e \rightarrow \nu_e} \simeq 1 - \frac{1}{2} \sin^2 2\theta_{12} \geq \frac{1}{2} \simeq 0.57.
\tag{5.71}
\]

Solar neutrino measurements also allow a determination of \(\Delta m^2_{12} \sim 8 \times 10^{-5} \text{ eV}^2\).

### 5.4.3 Oscillation Parameters

Here we summarize the known neutrino oscillation parameters. More details are readily available in the PDG [17]. Most of the numbers below are approximate, the PDG provides the best fit as well as uncertainties, as well as a wealth of different plots covering most of the aforementioned topics in great detail.

We have
\[
|U_{\text{PMNS}}| \sim 
\begin{pmatrix}
0.8 & 0.55 & 0.15 \\
0.4 & 0.6 & 0.7 \\
0.4 & 0.6 & 0.7
\end{pmatrix}
\tag{5.72}
\]

For the mass splittings,
\[
|\Delta m^2_{32}| = (\Delta m^2_{\text{atm}}) \simeq 2.40 \times 10^{-3} \text{ eV}^2
\tag{5.73}
\]
\[
\rightarrow |\Delta m_{32}| \approx 0.009 \text{ eV}
\tag{5.74}
\]
\[
\Delta m^2_{21} = 7.7 \times 10^{-5} \text{ eV}^2
\tag{5.75}
\]
\[
\rightarrow \Delta m_{21} \approx 0.049 \text{ eV}.
\tag{5.76}
\]

Note that while the sign of \(\Delta m_{32}\) is unknown, we do know the sign of \(\Delta m^2_{21}\), due to matter effects.
Figure 10: From [18]. Cartoon of the two possible neutrino mass hierarchies that fit the current data. The color shading indicates the fraction $|U_{\alpha i}|^2$ of each distinct flavor $\nu_\alpha$, $\alpha = e, \mu, \tau$ contained in each mass eigenstate $\nu_i$, $i = 1, 2, 3$. For example, $|U_{e2}|^2$ is equal to the fraction of the $(m_2)^2$ “bar” that is painted red (shading labeled as “$\nu_e$”). See the original paper for details.

Sum of Neutrino Masses

Anomalies

5.4.4 The Mass Hierarchy

As alluded to earlier, one of the most outstanding issues in neutrino physics is the undetermined mass hierarchy. Because we don’t know the sign of $\Delta m^2_{32}$, there are two possible hierarchies allowed: the so-called “normal” and “inverted” hierarchies. This is illustrated in the figure below, taken from the Snowmass Intensity Frontier Neutrino Working Group summary document [18].

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